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Stability of Stochastic Discrete Systems

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In this paper the stability of stochastic discrete systems will be discussed. Some results in this direction are given in [1]-[3].

1. DIFFERENT NOTIONS OF STABILITY FOR STOCHASTIC SYSTEMS

Let $\{\Omega, K, P\}$ be a probability field.

Denote by X the space of n-dimensional random vectors, $L^p(\Omega)$- the set of elements x from X which satisfies $E \mid x(\omega) \mid^p < \infty$ (where E denotes the mean value), and $L^\infty(\Omega)$ the set of elements from X which are essentially bounded.

If $x(\omega) \in L^p(\Omega)$ we denote $\| x(\omega) \|_p = (\int_\Omega \mid x(\omega) \mid^p P(d\omega))^{1/p}$ and if $x(\omega) \in L^\infty$, we denote

$$\| x \|_\infty = \inf_{\substack{A \in K \\ P(A)=0}} \sup_{\omega \in CA} \mid x(\omega) \mid .$$

Let $f_i : R^n \times \Omega \to R^n$ be a sequence of functions continuous in $x \in R^n$ and measurable with respect to $\omega \in \Omega$

Consider the system

(I) $\quad x_{n+1}(\omega) = f_n(x_n(\omega), \omega), \ \omega \in \Omega, \ n \in N,$

where N is the set of positive integers.

If $x_1(\omega)$ is measurable then $x_n(\omega)$ is a random vector for all $n \in N$.

For all $n_0 \in N$ and $x_0(\omega) \in X$ the system (I) has a unique solution (functions which coincide almost everywhere are considered equal) defined for $n \geqslant n_0$ (we consider that $x_{n_0}(\omega) = x_0(\omega)$). This solution will be denoted by $x_n(n_0, x_0, \omega)$.

Suppose that $f_n(0, \omega) = 0$.

Let $\mathscr{A}$ be a subset of X.

DEFINITION. (1) The trivial solution of system I is stable in probability

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with respect to $\mathscr{A}$ if for all $\epsilon > 0$, $\eta > 0$, $n_0 \in N$ there exists $\delta > 0$ such that, if $x_0(\omega) \in \mathscr{A}$ and $P\{\omega, | x_0(\omega) | > \delta\} < \delta$ then $P\{\omega, | x_n(n_0, x_0, \omega) | > \eta\} < \epsilon$ for $n \geqslant n_0$

(2) The trivial solution of system I is strongly stable in probability with respect to $\mathscr{A}$ if for all $\epsilon > 0$, $\eta > 0$, $n_0 \in N$ there exists $\delta > 0$ such that, if $x_0(\omega) \in \mathscr{A}$ and $P\{\omega, | x_0(\omega) | > \delta\} < \delta$, then

$$P\{\omega, \sup_{n \geqslant n_0} | x_n(n_0, x_0, \omega) | > \eta\} < \epsilon.$$

(3) The trivial solution of system I is stable in $L^p(\Omega)$ $1 \leqslant p \leqslant \infty$ with respect to $\mathscr{A}$ if for all $\epsilon > 0$, $n_0 \in N$ there exists $\delta > 0$ such that, if $x_0(\omega) \in \mathscr{A} \cap L^p(\Omega)$ and $\| x_0(\omega) \|_p < \delta$ then $\| x_n(n_0, x_0, \omega) \|_p < \epsilon$ for $n \geqslant n_0$.

(4) The trivial solution of system I is uniformly stable in probability with respect to $\mathscr{A}$, respectively, strongly uniformly stable in probability with respect to $\mathscr{A}$, uniformly stable in $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, with respect to $\mathscr{A}$, if (1), (2), (3) are satisfied, respectively, and δ does not depend on n_0.

(5) The trivial solution of system I is almost surely attractive with respect to $\mathscr{A}$ if $\lim_{n\to\infty} x_n(n_0, x_0, \omega) = 0$ almost everywhere (a.e.) for $n_0 \in N$, $x_0 \in \mathscr{A}$.

(6) The trivial solution of system I is attractive in probability with respect to $\mathscr{A}$ if $\lim_{n\to\infty P} x_n(n_0, x_0, \omega) = 0$ for $n_0 \in N$, $x_0 \in \mathscr{A}$ (where $\lim_P$ denotes the limit in probability).

(7) The trivial solution of system I is attractive in $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ with respect to $\mathscr{A}$ if $\lim_{n\to\infty} \| x_n(n_0, x_0, \omega) \|_p = 0$ for $n_0 \in N$, $x_0 \in \mathscr{A} \cap L^p(\Omega)$.

(8) The trivial solution of system I is asymptotically stable in probability with respect to $\mathscr{A}$ if it is stable and attractive in probability with respect to $\mathscr{A}$.

(9) The trivial solution of system I is asymptotically stable in $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, with respect to $\mathscr{A}$ if it is stable and attractive in $L^p(\Omega)$ with respect to $\mathscr{A}$.

(10) The trivial solution of system I is asymptotically stable in mean square if it is uniformly stable in $L^2(\Omega)$ with respect to $L^2(\Omega)$ and if the following property is satisfied: there exists $\delta_0 > 0$ and a function $n(\epsilon)$ such that if $\| x_0(\omega) \|_2 < \delta_0$ we have $\| x_n(n_0, x_0, \omega) \|_2 < \epsilon$ for $n - n_0 > n(\epsilon)$.

REMARK 1. It is easy to prove the following assertions:

(a) If the trivial solution is strongly stable in probability with respect to $\mathscr{A}$, then it is stable in probability with respect to $\mathscr{A}$.

(b) If the trivial solution is attractive in L^∞ with respect to $\mathscr{A}$, then it is attractive in $L^1(\Omega)$ with respect to $\mathscr{A}$. (This assertion follows from $\| x(\omega) \|_1 \leqslant \| x(\omega) \|_\infty$.)

(c) If the trivial solution is attractive in $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ with respect to $\mathscr{A}$ then it is attractive in probability with respect to $\mathscr{A}$.

(d) If the trivial solution is almost surely attractive with respect to $\mathscr{A}$ then it is attractive in probability with respect to $\mathscr{A}$.

(e) If $\mathscr{A} = \{x \in X, x(\omega) = x_0 \text{ a.e. } x_0 \in R^n\}$ then the stability in L^p, $1 \leqslant p \leqslant \infty$ with respect to $\mathscr{A}$ implies the stability in probability with respect to $\mathscr{A}$. Indeed, if $x(\omega) = x_0$ a.e. for $x_0 \in R^n$ then we have

$$\| x(\omega) \|_\infty \leqslant | x_0 | = \| x(\omega) \|_p = \| x(\omega) \|_1 \leqslant \| x(\omega) \|_\infty .$$

2. The Method of Liapunov Functions

Let $a(r)$, $b(r)$, $c(r)$ be continuous and increasing functions defined for $r \geqslant 0$, $a(0) = b(0) = c(0) = 0$, and $V_i : R^n \to [0, \infty)$, $V_i(0) = 0$, a sequence of continuous functions.

Let $S(M) = \{x, x \in L^\infty(\Omega), \| x(\omega) \|_\infty < M\}$.

THEOREM 1. *If*

(1) $a(| x |) \leqslant V_i(x)$, $i \in N$, $x \in R^n$

(2) $EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega))$

for $n_0 \in N$, $n \geqslant n_0$, $x_0 \in S(M)$, *then the trivial solution of system I is stable in probability with respect to* $S(M)$.

PROOF. Let $x_0(\omega) \in S(M)$. We have $| x_0(\omega) | < M$ a.e.

From condition (2) it follows that

$$EV_n(x_n(n_0, x_0, \omega)) \leqslant EV_{n_0}(x_0(\omega)).$$

Since V_{n_0} is a continuous function and $V_{n_0}(0) = 0$, $V_{n_0}(x) > 0$ for $x \neq 0$ there exists a continuous and increasing function $b_{n_0}(r)$ such that $V_{n_0}(x) \leqslant b_{n_0}(| x |)$ for all $x \in R^n$.

Hence

$$EV_n(x_n(n_0, x_0, \omega)) \leqslant Eb_{n_0}(| x_0(\omega) |).$$

Let $\epsilon > 0$, $\eta > 0$ and $\delta > 0$ such that

$$\frac{b_{n_0}(M)\,\delta + b_{n_0}(\delta)}{a(\eta)} < \epsilon.$$

Let

$$A = \{\omega, | x_0(\omega) | \geqslant \delta\}, \qquad \text{and} \qquad A_n = \{\omega, | x_n(n_0, x_0, \omega) | > \eta\}.$$

If $P(A) < \delta$ we have

$$Eb_{n_0}(|x_0(\omega)|) = \int_A b_{n_0}(|x_0(\omega)|)\,P(d\omega) + \int_{CA} b_{n_0}(|x_0(\omega)|)\,P(d\omega)$$
$$\leqslant b_{n_0}(M)\,P(A) + b_{n_0}(\delta) \leqslant b_{n_0}(M)\,\delta + b_{n_0}(\delta).$$

Hence

$$P(A_n)\,a(\eta) \leqslant \int_{A_n} V_n(x_n(n_0\,, x_0\,, \omega))\,P(d\omega) \leqslant b_{n_0}(M)\,\delta + b_{n_0}(\delta).$$

Thus $P(A_n) < \epsilon$ and the theorem is proved.

REMARK 2. If in Theorem 1 we suppose, in addition, that $V_n(x) \leqslant b(|x|)$, then the trivial solution of system I is uniformly stable in probability with respect to $S(M)$.

THEOREM 2. *If*

(1) $a(|x|) \leqslant V_i(x),\ \ i \in N,\ \ x \in R^n$

(2) *For all* $n_0 \in N$, $n \geqslant n_0$, $x_0(\omega) \in S(M)$, $V_n(x_n(n_0\,, x_0\,, \omega))$ *is a supermartingale, i.e.,*

$$E[V_{n+1}(x_{n+1}(n_0\,, x_0\,, \omega)) \mid V_{n_0}(x_0(\omega)),\dots, V_n(x_n(n_0\,, x_0\,, \omega))]$$
$$\leqslant V_n(x_n(n_0\,, x_0\,, \omega)) \qquad \text{a.e.},$$

then the trivial solution of system I is strongly in probability with respect to $S(M)$.

PROOF. Let $x_0(\omega) \in S(M)$.

From Condition (2) it follows that $EV_n(x_n(n_0\,, x_0\,, \omega)) \leqslant EV_{n_0}(x_0(\omega))$.

There exists a continuous and increasing function $b_{n_0}(r)$, such that $V_{n_0}(x) \leqslant b_{n_0}(|x|)$.

Let $\epsilon > 0$, $\eta > 0$ and $\delta > 0$ such that

$$\frac{1}{a(\eta)}\,(b_{n_0}(M)\,\delta + b_{n_0}(\delta)) < \frac{\epsilon}{2}\,.$$

Let

$A = \{\omega, |x_0(\omega)| > \delta\}$;

$B = \{\omega, \sup\limits_{n \geqslant n_0} |x_n(n_0\,, x_0\,, \omega)| > \eta\} = \{\omega, \sup\limits_{n \geqslant n_0} a(|x_n(n_0\,, x_0\,, \omega)|) > a(\eta)\}\,.$

If $P(A) < \delta$ we have $EV_{n_0}(x_0(\omega)) \leqslant Eb_{n_0}(|x_0(\omega)|) \leqslant b_{n_0}(M)\,\delta + b_{n_0}(\delta)$.

Since $V_n(x_n(n_0\,, x_0\,, \omega))$ is a supermartingale, we have [4]

$$a(\eta)\,P(S_n) \leqslant EV_{n_0}(x_0(\omega)) + E\,|V_n(x_n(n_0\,, x_0\,, \omega))| \leqslant 2Eb_{n_0}(|x_0(\omega)|),$$

where

$$S_n = \{\omega, \max_{n_0 \leqslant j \leqslant n} V_n(x_n(n_0, x_0, \omega)) \geqslant a(\eta)\}.$$

Let

$$S = \lim_{n\to\infty} S_n = \bigcup_{n=n_0}^{\infty} S_n .$$

We have

$$a(\eta)\, P(S) \leqslant 2Eb_{n_0}(|\, x_0(\omega)\,|).$$

Since $B \subset S$ we have

$$a(\eta)\, P(B) \leqslant a(\eta)\, P(S) \leqslant 2(b_{n_0}(M)\, \delta + b_{n_0}(\delta)).$$

Hence $P(B) < \epsilon$ and Theorem 2 is proved.

REMARK 3. If in Theorem 2 we suppose, in addition, that $V_n(x) \leqslant b(|\, x\,|)$, then the trivial solution of system I is strongly uniformly stable in probability with respect to $S(M)$.

THEOREM 3. *If*

$$EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega)) - Ec(|\, x_n(n_0, x_0, \omega)\,|)$$

for all $n_0 \in N$, $n \geqslant n_0$, $x_0(\omega) \in \mathscr{A}$, *and* $EV_{n_0}(x_0(\omega)) < \infty$, *then the trivial solution of system I is almost surely attractive with respect to* $\mathscr{A} \cap L^1(\Omega)$.

PROOF. Let $x_0(\omega) \in \mathscr{A} \cap L^1(\Omega)$ and $z_n(\omega) = V_n(x_n(n_0, x_0, \omega))$.
We have

$$\sum_{i=n_0}^{n-1} E[z_{i+1}(\omega) - z_i(\omega)] \leqslant -\sum_{i=n_0}^{n-1} Ec(|\, x_i(n_0, x_0, \omega)\,|).$$

Hence

$$\sum_{i=n_0}^{n-1} Ec(|\, x_i(n_0, x_0, \omega)\,|) \leqslant Ez_{n_0}(\omega) - Ez_n(\omega) \leqslant Ez_{n_0}(\omega) < \infty.$$

Thus

$$\sum_{n=n_0}^{\infty} Ec(|\, x_n(n_0, x_0, \omega)\,|) < \infty.$$

By Fatou's lemma it follows that the series $\sum_{n=n_0}^{\infty} c(|\, x_n(n_0, x_0, \omega)\,|)$ is convergent almost everywhere.
Hence $\lim_{n\to\infty} c(|\, x_n(n_0, x_0, \omega)\,|) = 0$.
Thus, $\lim_{n\to\infty} x_n(n_0, x_0, \omega) = 0$ a.e., and theorem is proved.

THEOREM 4. *If*

(1) $\beta \mid x \mid^p \leqslant V_n(x) \leqslant b(\mid x \mid)$, *where* $\beta > 0$,

(2) $EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega)) - Ec(\mid x_n(n_0, x_0, \omega) \mid)$ *for all* $n_0 \in N$, $n \geqslant n_0$, $x_0(\omega) \in \mathscr{A}$, *then the trivial solution of system I is almost surely attractive with respect to* $\mathscr{A}$ *and also is attractive in* L^p *with respect to* $\mathscr{A}$.

PROOF. From Theorem 3 it follows that the trivial solution of system I is almost surely attractive with respect to $\mathscr{A}$.

Let $z_n(\omega) = V_n(x_n(n_0, x_0, \omega))$.

From Condition (2) it follows that $Ez_n(\omega)$ is a decreasing sequence. Let $\gamma = \lim_{n\to\infty} Ez_n(\omega)$. Hence $\lim_{n\to\infty P} z_n(\omega) = \gamma$. By Riesz's theorem there exists a subsequence $z_{n_i}(\omega)$ such that $\lim_{i\to\infty} z_{n_i}(\omega) = \gamma$ a.e.

From the inequality $z_{n_i}(\omega) \leqslant b(\mid x_{n_i}(n_0, x_0, \omega) \mid)$ it follows that $\gamma = 0$. Since

$$E \mid x_n(n_0, x_0, \omega) \mid^p \leqslant \frac{1}{\beta} Ez_n(\omega).$$

We have $\lim_{n\to\infty} \| x_n(n_0, x_0, \omega) \|_p = 0$, and thus Theorem 4 is proved.

THEOREM 5. *If*

(1) $a(\mid x \mid) \leqslant V_n(x)$

(2) $EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega)) - Ec(\mid x_n(n_0, x_0, \omega)\mid)$ *for* $n_0 \in N$, $n \geqslant n_0$, $x_0(\omega) \in S(M)$,

then the trivial solution of system I is asymptotically stable in probability with respect to $S(M)$.

PROOF. From Theorem 1 it follows that the trivial solution of system I is stable in probability with respect to $S(M)$.

From Theorem 3 it follows that the trivial solution of system I is almost surely attractive with respect to $S(M)$ and hence it is attractive in probability with respect to $S(M)$.

THEOREM 6. *If*

(1) $\alpha \mid x \mid^2 \leqslant V_n(x) \leqslant \beta \mid x \mid^2$, $\alpha > 0$, $\beta > 0$

(2) $EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega))$

for $n_0 \in N$, $n \geqslant n_0$, $\| x_0 \|_2 \leqslant H$,

then the trivial solution of system I is uniformly stable in $L^2(\Omega)$ *with respect to*

$\mathscr{A} = \{x, x \in L^2(\Omega), \| x \|_2 \leqslant H\}$.

PROOF. Let $\epsilon > 0$ and $\delta(\epsilon) = (\alpha/\beta)\,\epsilon$. If $\| x_0(\omega) \|_2^2 < \delta(\epsilon)$ we have

$$\| x_n(n_0, x_0, \omega) \|_2^2 \leqslant \frac{1}{\alpha} EV_n(x_n(n_0, x_0, \omega)) \leqslant \frac{1}{\alpha} EV_{n_0}(x_0(\omega))$$

$$\leqslant \frac{\beta}{\alpha} \| x_0(\omega) \|_2^2 < \epsilon.$$

Thus Theorem 6 is proved.

THEOREM 7. *If*

(1) $\alpha \mid x \mid^2 \leqslant V_n(x) \leqslant \beta \mid x \mid^2$

(2) $EV_{n+1}(x_{n+1}(n_0, x_0, \omega)) \leqslant EV_n(x_n(n_0, x_0, \omega)) - \gamma E \mid x_n(n_0, x_0, \omega) \mid^2$ *for* $n_0 \in N$, $n \geqslant n_0$, $\| x_0(\omega) \|_2 \leqslant H$ *and where* $\gamma > 0$,

then the trivial solution of system I is asymptotically stable in mean square.

PROOF. Let $\epsilon > 0$ and $\delta_0 > 0$. From Theorem 6 it follows that, if $\| x_0(\omega) \|_2^2 < (\alpha/\beta)\,\epsilon$ we have $\| x_n(n_0, x_0, \omega) \|_2^2 < \epsilon$ for $n \geqslant n_0$.

Let $n(\epsilon)$ such that $n(\epsilon) > (\beta^2\delta_0/\gamma\alpha\epsilon)$.

If $\| x_0(\omega) \|_2^2 < \delta_0$ there exists $n_1 \in N$, $n_0 \leqslant n_1 < n_0 + n(\epsilon)$ such that

$$\| x_{n_1}(n_0, x_0, \omega) \|_2^2 < \frac{\alpha}{\beta}\,\epsilon.$$

Indeed, suppose that for all $n \in N$, $n_0 \leqslant n < n_0 + n(\epsilon)$ we have

$$\| x_n(n_0, x_0, \omega) \|_2^2 \geqslant \frac{\alpha}{\beta}\,\epsilon.$$

From condition (2) it follows that

$$\sum_{n=n_0}^{n_0+n(\epsilon)} E[V_{n+1}(x_{n+1}(n_0, x_0, \omega)) - V_n(x_n(n_0, x_0, \omega))]$$

$$\leqslant -\gamma \sum_{n=n_0}^{n_0+n(\epsilon)} E \mid x_n(n_0, x_0, \omega) \mid^2.$$

Hence

$$EV_{n_0+n(\epsilon)}(x_{n_0+n(\epsilon)}(n_0, x_0, \omega)) \leqslant EV_{n_0}(x_0(\omega)) - \gamma n(\epsilon) \frac{\alpha}{\beta}\,\epsilon$$

$$< \beta\delta_0 - \gamma n(\epsilon) \frac{\alpha}{\beta}\,\epsilon < 0,$$

and thus we get a contradiction.

Hence, there exists $n_1 \in N$, $n_0 \leqslant n_1 < n_0 + n(\epsilon)$ such that

$$\| x_{n_1}(n_0, x_0, \omega) \|_2^2 < \frac{\alpha}{\beta} \epsilon.$$

Thus, for $n > n_0 + n(\epsilon)$ we have

$$\| x_n(n_0, x_0, \omega) \|_2^2 = \| x_n(n_1, x_{n_1}(n_0, x_0, \omega), \omega) \|_2^2 < \epsilon,$$

and theorem is proved.

EXAMPLE. Consider the system

$$\begin{cases} x_{n+1}(\omega) = \alpha_n(\omega)\, x_n(\omega) + \beta_n(\omega)\, y_n(\omega) \\ y_{n+1}(\omega) = \gamma_n(\omega)\, x_n(\omega) + \delta_n(\omega)\, y_n(\omega). \end{cases}$$

Suppose that $\{\alpha_n(\omega), \beta_n(\omega), \gamma_n(\omega), \delta_n(\omega)\}$ are independent, and

$$E[\alpha_n(\omega)\, \beta_n(\omega) + \gamma_n(\omega)\, \delta_n(\omega)] = 0,$$

$$E(\alpha_n^2(\omega) + \gamma_n^2(\omega)) < 1, \quad E(\beta_n^2(\omega) + \delta_n^2(\omega)) < 1.$$

Let $\mathscr{A}$ be the set of bidimensional random vector, $x_0(\omega)$ with the propriety that $\{x_0(\omega), \alpha_n(\omega), \beta_n(\omega), \gamma_n(\omega), \delta_n(\omega)\}$ are independent.

Let $V(x, y) = x^2 + y^2$.

We have

$$E[x_{n+1}^2(\omega) + y_{n+1}^2(\omega) \mid x_1, y_1, \dots, x_n, y_n]$$

$$= x_n^2(\omega)\, \{E[\alpha_n^2(\omega) \mid x_1, y_1, \dots, x_n, y_n] + E[\gamma_n^2(\omega) \mid x_1, y_1, \dots, x_n, y_n]\}$$

$$+ y_n^2(\omega)\, E[\beta_n^2(\omega) + \delta_n^2(\omega) \mid x_1, y_1, \dots, x_n, y_n]$$

$$+ x_n(\omega)\, y_n(\omega)\, E[\alpha_n(\omega)\, \beta_n(\omega) + \gamma_n(\omega)\, \delta_n(\omega) \mid x_1, y_1, \dots, x_n, y_n]$$

$$= x_n^2(\omega)\, E(\alpha_n^2(\omega) + \gamma_n^2(\omega)) + y_n^2(\omega)\, E(\beta_n^2(\omega) + \delta_n^2(\omega)).$$

Hence

$$E[x_{n+1}^2(\omega) + y_{n+1}^2(\omega) \mid x_1, y_1, \dots, x_n, y_n] \leqslant x_n^2(\omega) + y_n^2(\omega).$$

Thus

$$E[V(x_{n+1}(\omega), y_{n+1}(\omega)) \mid V(x_1, y_1), \dots, V(x_n, y_n)]$$

$$= E\{E[x_{n+1}^2(\omega) + y_{n+1}^2(\omega) \mid x_1, y_1, \dots, x_n, y_n] \mid V(x_1, y_1), \dots, V(x_n, y_n)\}$$

$$\leqslant E[x_n^2(\omega) + y_n^2(\omega) \mid V(x_1, y_1), \dots, V(x_n, y_n)] = V(x_n, y_n).$$

Hence, from Remark 3 it follows that the trivial solution of system II is strongly uniformly stable in probability with respect to $\mathscr{A} \cap S(M)$ and from Theorem 6 it follows that it is uniformly stable in $L^2(\Omega)$ with respect to $\mathscr{A} \cap L^2(\Omega)$.

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